Asset Markets

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Tonight's Lecture

- Varian two-period model
- Equilibrium under certainty
- Mean-variance
- The CAPM
- Arbitrage Price Theory and Expected Utility

Tonight's Lecture

You were to work through the two-period model in Varian's Microeconomic Analysis (the graduate text) on Pages 184 to 186.

- What are the key assumptions of this model?
- Describe the approach Varian uses to solve the model.
- Can you interpret the optimality conditions?
- What are the results of this model?
- What are the testable predictions of this model?
- How does Varian produce these testable predictions from the model?
- How would you take these testable predictions to the data?

Reading

- Hal Varian, Intermediate Microeconomics Chapter 11 (Asset Markets)
- Hal Varian, Intermediate Microeconomics Chapter 13 (Risky Assets)
- Hal Varian, Microeconomic Analysis Chapter 20 (Asset Markets)
- Advanced reading: Yvan Lengwiler, Microfoundations of Financial Economics Chapter 5 (Static Finance Economy)

Assets

- Assets can broadly be defined as a good that provides of flow of services over time.
- Assets that provide a monetary flow are called financial assets.
- Financial assets are contracts that deliver a state-dependent amount of money in the future.
- Financial assets are usually themselves a claim the flow of services from some physical asset.

Assets under certainty

- In a world where the returns on assets are certain, then the price of an asset must be the present discounted value of its stream of returns and the rate of return must be the same for all assets.
- This is due to the simple concept of **arbitrage**.
- Suppose there are two assets that pay sure returns in a two period model.
- Suppose we know that the first asset pays a rate of return of r_0.
- How would we value the second asset?

Arbitrage under certainty

- If the second asset a will have a value V_a in the second period, then what is the equilibrium price you should pay for the asset today?
- Pick any arbitrary price p_a.
- Suppose you sold asset a today at the price p_a, invested it in the first asset and got a total return $p_a(1 + r_0)$ tomorrow.
- If $p_a(1 + r_0) > V_a$, then you would have made a profit by selling asset a (compared to holding onto it).
- Since it is profitable to sell asset a, then at least one person will want to sell asset a at price p_a and no reasonable person would choose to buy asset a at price p_a.
- This means the price of asset a is too high, and by the laws of supply and demand, the price p_a must fall.

Equilibrium price under certainty

- The opposite situation would hold if p_a were too low so that $p_a(1 + r_0) < V_a$—at least one person would want to buy the asset and no one would want to sell.
- Equilibrium is then defined where:

$$p_a = \frac{V_a}{1 + r_0}$$

or equivalently:

$$r_a = \frac{V_a}{p_a} - 1 = r_0.$$

- **Testable prediction:** Under certainty, no one would hold an asset that is going to have a lower return than another asset.

Assets under uncertainty

- When there is uncertainty about the return on an asset, one way to describe the probability distribution of potential outcomes is to use summary statistics: **moments**.
- The first two moments of a distribution—the mean and variance—simplify the description of choice under uncertainty significantly.
- For a set of possible states of nature $s = 1 \dots S$ where each occurs with probability π_s and the outcome in each state defined as w_s, then the mean is:

$$\mu_w = \sum_{s=1}^{S} \pi_s w_s$$

and the variance is:

$$\sigma_w^2 = \sum_{s=1}^{S} \pi_s (w_s - \mu_w)^2$$

Mean-Variance

- We can express our utility function as $u(\mu_w, \sigma_w^2)$ or $u(\mu_w, \sigma_w)$.
- This mean-variance approach is a simplification of the expected utility framework and therefore gives us a rough approximation of risk aversion.
- The basic principle continues to hold: expected consumption is a good (positive utility) and more variation in consumption is a bad (negative utility).
- The mean-variance framework should rank options in the same way as a more complete expected utility framework, but only if the choices can be entirely characterised by the mean and the variance.

Calculating the Mean and Variance

- The simplest setting is where there are two assets: a risk-free asset that pays a sure return of r_f, and a risky asset with an expected return of r_m and a variance of σ_m^2.
- Beginning with wealth w, you can choose to invest a fraction x in the risky asset and the remaining share $(1-x)$ in the risk-free asset.
- For any share x, the individual's mean return will be:

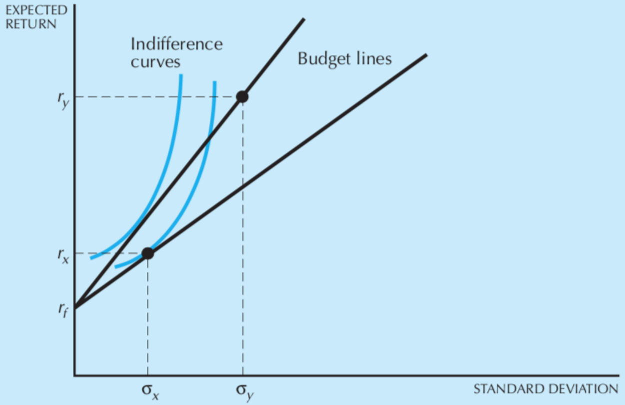
$$r_x = xr_m + (1-x)r_f$$

  and the variance will be:

$$\sigma_w^2 = x^2\sigma_m^2$$

- Note that $\sigma_f^2 = 0$.

Graphical Optimal Choice



Preferences between risk and return. The asset with risk-return combination y is preferred to the one with combination x.

Pricing Risk

- The previous graph plots the options that the investor has: getting more return requires you to take on more risk.
- From the previous graph, we can define a move up/right along the budget line as increasing the share invested in the risky asset x by a little bit.
- Intuitively this means that we are receiving an extra $(r_m - r_f)$ return for every extra unit of risk we take on:

$$\frac{r_m - r_f}{\sigma_m}.$$

The CAPM Model

- Let's consider a two-period model with many risky assets.
- An investor i has wealth W_i in period 1 and is investing in either a risk-free asset or any of many risky assets.
- All assets are denoted by $a = 0 \dots A$ where a_0 is the risk-free asset. This means assets $a = 1 \dots A$ are risky.
- If an investor invests in risky assets in period 1, then period 2 consumption is uncertain, and can be written as:

$$\tilde{c}_2 = (W - c_1)\left[x_0 R_0 + \sum_{a=1}^{A} x_a \tilde{R}_a\right] = (W - c_1)\left[R_0 + \sum_{a=1}^{A} x_a(\tilde{R}_a - R_0)\right]$$

using the definition of the weights $x_0 = 1 - \sum_{a=1}^{A} x_a$ to simplify.

Mean-Variance Efficiency

- The term $R_0 + \sum_{a=1}^{A} x_a(\tilde{R}_a - R_0)$ is the gross expected portfolio return.
- For any expected return (mean) the investor would like to have as little risk (variance) as possible on that return.
- That is, the investor wants to purchase a portfolio that is **mean-variance efficient**.
- That is, for any given expected return, the investor's optimal portfolio should minimize variance.

Minimizing Variance: the problem

- Variance in a world with multiple risky assets is defined as:

$$\sum_{a=0}^{A} \sum_{b=0}^{A} x_a x_b \sigma_{ab}$$

  where σ_{ab} is now the covariance between the returns on two assets a and b, or $\text{cov}(\tilde{R}_a, \tilde{R}_b)$. Note that $\sigma_{aa} = \sigma_a^2$.

- We now want choose the set of weights $x_0, \ldots, x_A$ that will minimize the variance such that we achieve an arbitrary level of return:

$$\bar{R} = \sum_{a_0}^{A} x_a R_a$$

  and the weights of course need to sum to 1: $\sum_{a=0}^{A} x_a = 1$.

Minimizing Variance: the solution

- The Lagrange of this problem can be written as:

$$L = \sum_{a=0}^{A}\sum_{b=0}^{A} x_a x_b \sigma_{ab} + \lambda(\bar{R} - \sum_{a_0}^{A} x_a R_a) + \mu(1 - \sum_{a=0}^{A} x_a)$$

- With the first order conditions:

$$2\sum_{b=0}^{A} x_b \sigma_{ab} - \lambda R_a - \mu = 0, \qquad \text{for } a = 0, \ldots, A$$

A Mutual Fund

- Let's assume there's some portfolio e made up entirely of risky assets that satisfies these conditions, where the shares of this portfolio are denoted $(x_1^e, \ldots, x_A^e)$.
- Imagine some mutual fund holds this efficient portfolio, and then issues a risky asset that is effectively made up of this efficient portfolio.
- This risky asset is available to individual investors as asset e.
- If an individual invests their entire portfolio in asset e, then they are holding a mean-variance efficient portfolio.

A Mutual Fund

- Since we're only investing in asset e, then $x_b = 0$ as long as $b \neq e$.
- This means the a^{th} FOC becomes:

$$2(1)\sigma_{ae} - \lambda R_a - \mu = 0.$$

- For $a = 0$ (the risk-free asset):

$$-\lambda R_0 - \mu = 0$$

- And for the mutual fund portfolio $a = e$:

$$2\sigma_{ee}^2 - R_e - \mu = 0$$

A Mutual Fund

- For given R_a, we simply need to solve these two FOCs for λ and μ as simultaneous equations, giving:

$$\lambda = \frac{2\sigma_{ee}}{R_e - R_0}$$
$$\mu = -\lambda R_0 = -\frac{2\sigma_{ee} R - 0}{R_e - R_0}$$

- Substituting into the a^{th} FOC and rearranging:

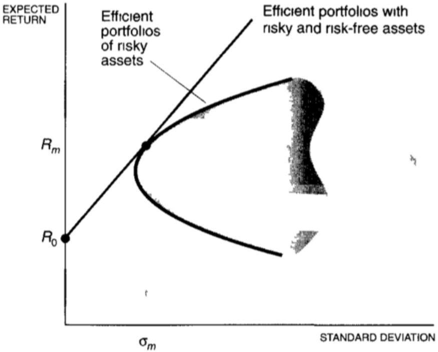
$$R_a = R_0 + \frac{\sigma_{ae}}{\sigma_{ee}}(R_e - R_0)$$

- The expected return on any asset is equal to the risk free rate plus some **risk premium** that depends on how the asset covaries with the market portfolio.

An Efficient Portfolio of Risky Assets

- So we've assumed the existence of an efficient portfolio. Where do we find one?
- Let's look at the set of portfolios that use only risky assets.
- Plotting them on a chart of risk v return, we know that of the set of possible *risky* portfolios, for any given return, the one that minimizes the variance if the portfolio that is furthest to the left.
- Now adding the risk-free asset, we want the set of linear combinations of the risk-free asset and the portfolio of risky assets that are efficient.

An Efficient Portfolio of Risky and Risk-free Assets



Set of risky returns and standard deviations. All mean-variance efficient portfolios can be constructed by combining the portfolios whose returns are R_0 and $\overline{R}_m$.

An Efficient Portfolio of Risky and Risk-free Assets

- This efficient portfolio of risky and risk-free assets can be found by combining the risk-free asset with the set of efficient risky portfolios by a line.
- We want the line that gives us the highest slope, where the slope defines the ratio of expected return to risk.
- Therefore, we find that portfolio of risky assets m as the point where the line from the risk-free asset is just tangential to the set of efficient risky portfolios.

The Efficient Portfolio m

- What do we know about this specific efficient portfolio m?
- Assume that x_a^m is the share of wealth invested in asset a in portfolio m.
- Assume that an individual invests some amount of wealth W_i in portfolio m.
- Assume that p_a is the price of asset a and X_{ia} is the number of shares individual i holds of asset a.
- Note that the choice of portfolio m has not relied on an individual's preferences *at all*.
- This means all individuals hold the same portfolio of risky assets m.

Market Portfolio of Risky Assets

- Then we can say that for all individuals i, we must have:

$$x_a^m = \frac{p_a X_{ia}}{W_i}.$$

- Summing over i:

$$x_a^m = \frac{p_a \sum_i X_{ia}}{\sum_i W_i}.$$

- The numerator is the market value of asset a, while the denominator is the total value of all risky assets.
- Then x_a^m is simply the share of wealth invested in risky assets that is invested in asset a.
- This portfolio is therefore simply called the **market portfolio of risky assets**: you simply hold the aggregate portfolio!

The CAPM Result

- Simply plugging the market portfolio of risky assets (which is a mean-variance efficient portfolio) into our result from the mutual funds example, we get:

$$R_a = R_0 + \frac{\sigma_{am}}{\sigma_{mm}}(R_m - R_0)$$

- This says that the risk premium is the covariance of the return on asset a with the return on the market portfolio m divided by the variance of the return on the market portfolio, multiplied by the excess return on the market portfolio.
- You might notice that this $\text{cov}(R_a, R_m)/\text{var}(R_m)$ is simply the β_a you would get from a regression of R_a on R_m.

The CAPM Result

- This result is colloquially written as:

$$R_a = R_0 + \beta_a(R_m - R_0)$$

- **Testable prediction:** The Capital Asset Pricing Model says that in order to know the risk premium on an asset, we only need to know the covariance of the asset's return with the market portfolio.
- It is critical to note that only the *covariance* of the asset with the market matters, not the absolute variance of the asset's returns.
- This is because we only care how the asset contributes to the risk of the market portfolio *at the margin*, not in total.

Arbitrage Pricing Theory

- Arbitrage pricing theory starts from the supply side, rather than the demand for assets as CAPM does.
- The idea behind APT is that there is strong co-movement among asset prices, which is being driven by common factors.
- For example, assuming two ‘macroeconomic” or economy-wide factors that influence all asset returns, f_1 and f_2, we can write the price of asset a as:

$$R_a = b_{0a} + b_{1a}f_1 + b_{2a}f_2 + \epsilon_a$$

- Each asset is influenced differently based on the value of the b's, which are the sensitivity factors.
- Each asset also has its own constant (b_{0a}) and asset-specific risk (ϵ_a) with expectation zero.

Arbitrage Pricing Theory

Assume that there is no asset-specific risk. Then we can construct a portfolio with three assets a, b, and c, where:

$$
\begin{aligned}
x_a b_{1a} + x_b b_{1b} + x_c b_{1c} &= 0 \\
x_a b_{2a} + x_b b_{2b} + x_c b_{2c} &= 0 \\
x_a + x_b + x_c &= 1
\end{aligned}
$$

This portfolio eliminates the risk from factor 1 and factor 2, with the sum of asset shares equal to 1. If the portfolio eliminates all risk, then it must earn the riskless rate of return, implying:

$$R_a - R_0 = b_{1a}\lambda_1 + b_{2a}\lambda_2$$

where the λ's represent the excess returns on portfolios that have sensitivity $b = 1$ to a specific type of risk. **Testable prediction:** the excess return on asset a depends on its sensitivity to the two risky factors.

Expected Utility

- What if we began from expected utility rather than simple mean-variance utility?
- Consider the problem:

$$\max_{c_1, x_1, \ldots, x_A} \quad u(c_1) + \alpha E \left[u \left((W - c_1) \left(R_0 + \sum_{a=1}^{A} x_a (\tilde{R}_a - R_0) \right) \right) \right]$$

- The FOCs of this problem are:

$$u'(c_1) = \alpha E u'(c_2) \tilde{R}$$
$$0 = E u'(c_2)(\tilde{R}_a - R_0) \quad \text{for } a = 1, \ldots, A$$

which are quite similar to what we derived last week.

- **Testable prediction:** The final asset pricing equation from this expected utility model strongly resembles the CAPM:

$$R_a = R_0 + \frac{\sigma_{ca}}{\sigma_{cc}}(R_c - R_0)$$

where c is aggregate period 2 consumption and this ratio of covariances is called the **consumption beta** of an asset.

- **Testable prediction:** The expected utility model can also give us:

$$R_a - R_0 = -\frac{1}{Eu'(c_1)}\left[b_{1a}\text{cov}(u'(c_2), f_1) + b_{2a}\text{cov}(u'(c_2), f_2)\right]$$

implying that the λ's in our APT model are proportional to the covariance between the marginal utility of consumption and the relevant risk factor.

Arrow-Debreu Securities

- These are hypothetical securities that pay a return of $1 if a state of nature s occurs and zero otherwise.
- With some manipulation, these allow us to show that:

$$p_a = \frac{V_a}{R_0} + \text{cov}(F(C), V_a)$$

  where C is aggregate consumption and $F(C)$ is a *decreasing* function of C.
- **Testable prediction:** This says that an asset that positively covaries with aggregate consumption will have a negative adjustment factor while assets that negatively covary with consumption will have a positive adjustment factor.

Testable Predictions

- Each of these models have assumptions underlying them, some more restrictive than others.
- Each of these models provides some testable predictions about asset pricing.
- That is, these models aren't simply creative exercises, but aim to understand and explain asset prices.
- We use models to structure our thinking about how financial decisions are made, and there how financial markets work.

The End.